

# On a Conjecture of Micha Perles

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## **Abstract**

We prove a conjecture of Micha Perles concerning simple polytopes, for a subclass that properly contains the duals of stacked and crosspolytopes. As a consequence of a special property of this subclass it also follows that, the entire combinatorial structure of a polytope in the subclass can be recovered from its graph, by applying our results recursively.

# 1 Introduction

Let  $P$  be a simple  $d$ -polytope and  $G(P)$  the graph (1-skeleton) of  $P$ . Perles conjectured that every  $(d - 1)$ -regular, induced, connected and non-separating subgraph of  $G(P)$  determines a facet of  $P$  [2]. In this paper we prove the conjecture for a proper subclass of simple polytopes.

The motivation for our results comes from two subclasses of simplicial polytopes, namely the *stacked polytopes* and the *crosspolytopes*. Polytopes obtained from a simplex by successive addition of pyramids over facets are called *stacked polytopes*. Stacked polytopes form an important subclass of simplicial polytopes in that, only they attain equality in the lower bound theorem [3]. Secondly, if  $\{e_1, \dots, e_d\}$  is a set of linearly independent vectors in  $R^d$  then  $X = \text{conv}\{\pm e_1, \dots, \pm e_d\}$  is called a *d-crosspolytope*.  $d$ -crosspolytopes can be formed by successively building bipyramids  $d - 1$  times starting with a 1-simplex.

Consider the dual analogues of the operations of stacking and forming bipyramids. It can be shown [3] that, if  $P$  is a simplicial polytope and  $P^*$  a simple polytope dual to  $P$ , then a polytope obtained by forming a pyramid over a facet of  $P$  is dual to a polytope obtained by truncating the corresponding vertex of  $P^*$ . Also, [4] any bipyramid with basis  $P$  is dual to any prism with basis  $P^*$ .

Against this background, we show that if  $P$  is any polytope for which Perles' conjecture is true, then the conjecture is also true for a polytope obtained by truncating a vertex of  $P$  and also for any prism with basis  $P$ . Since Perles' conjecture is trivially true for a simplex, one concludes that it is true for any polytope obtained from a simplex by building prisms and truncating vertices finitely many times, in any arbitrary order. The class  $\mathcal{C}$  of polytopes so generated, properly contains duals of crosspolytopes and duals of stacked polytopes. We also show that  $\mathcal{C}$  is a proper subclass of simple polytopes. Moreover the class  $\mathcal{C}$  has the interesting property that every face of a polytope in  $\mathcal{C}$  also belongs to  $\mathcal{C}$ . This allows one to recover the entire combinatorial structure of a polytope in  $\mathcal{C}$  from its graph, by applying our results

recursively.

## 2 Notation

Please refer to [3] for a discussion on polytopes and for related terminology. Here we merely indicate the convention that we will follow in the sequel.

If  $P$  is a polytope then its vertex set will be denoted  $V(P)$  and its graph  $G(P)$ . The vertex set and the edge set of a graph  $\Gamma$  will be denoted  $V(\Gamma)$  and  $E(\Gamma)$  respectively. Given a  $d$ -polytope  $P$ , let  $\tilde{P} = P + \mathbf{z}$  be a translate of  $P$  where  $\mathbf{z}$  is a non-zero vector in  $R^{d+1}$ . Then the convex hull of  $P$  and  $\tilde{P}$  is called a *prism with basis  $P$*  denoted  $Pr(P)$ . A hyperplane  $H$  in  $R^d$  is said to truncate a vertex  $x$  of  $P$ , if  $x$  and  $V(P) \setminus \{x\}$  lie in different open half-spaces of  $H$ . We denote by  $Tr(P)$  the intersection of  $P$  with the closed half-space containing  $V(P) \setminus \{x\}$ . For our purposes, it does not matter which vertex of  $P$  is truncated to obtain  $Tr(P)$ .  $Tr(P)$  represents a (not necessarily unique) polytope obtained by truncating some vertex of  $P$ .

Define a subclass  $\mathcal{C}$  of simple polytopes as follows: *A polytope  $P$  belongs to  $\mathcal{C}$  iff there is a sequence of polytopes*

$$P_0, P_1, \dots, P_n = P$$

*where  $P_0$  is a  $k$ -simplex ( $k \geq 1$ ) and for  $1 \leq i \leq n$  either  $P_i = Tr(P_{i-1})$  or  $P_i = Pr(P_{i-1})$ .*

From the definition of the subclass  $\mathcal{C}$  it follows that it contains the duals of stacked and crosspolytopes.

## 3 Perles' Conjecture for the Subclass $\mathcal{C}$

**Theorem 1** *If Perles' conjecture is true for a simple  $d$ -polytope  $P$  then it is also true for any prism  $Pr(P)$  with basis  $P$ .*

**Proof :** Let  $Pr(P)$  be the convex hull of  $P$  and its translate  $\tilde{P} = P + \mathbf{z}$ . Then every vertex  $v \in V(P)$  is adjacent to the vertex  $\tilde{v} = v + \mathbf{z}$  of  $\tilde{P}$ . If  $X \subseteq V(P)$  then  $\tilde{X}$  will denote the corresponding subset of  $V(\tilde{P})$ . If  $\Lambda$  is any induced subgraph of  $P$  then  $\tilde{\Lambda}$  will denote the subgraph of  $G(\tilde{P})$  induced by the corresponding vertices of  $\tilde{P}$ .

Let  $\Gamma$  be a  $d$ -regular, induced, connected and non-separating subgraph of  $Pr(P)$ . We show that  $\Gamma$  must determine a facet of  $Pr(P)$ .

If  $\Gamma$  is a subgraph of  $G(P)$  (resp.  $G(\tilde{P})$ ), since both  $\Gamma$  and  $G(P)$  (resp.  $G(\tilde{P})$ ) are  $d$ -regular graphs,  $\Gamma = G(P)$  (resp.  $\Gamma = G(\tilde{P})$ ). Hence  $\Gamma$  determines a facet of  $Pr(P)$ . So we may assume that  $V(\Gamma) \cap V(P) \neq \emptyset$  and  $V(\Gamma) \cap V(\tilde{P}) \neq \emptyset$ .

Let  $\Gamma_P$  and  $\Gamma_{\tilde{P}}$  be the restrictions of  $\Gamma$  to  $P$  and  $\tilde{P}$  respectively. Consider any vertex  $v$  of  $\Gamma_P$ .  $v$  is adjacent to only one vertex in  $\tilde{P}$ . Also,  $\Gamma$  is  $d$ -regular. Hence  $v$  has at least  $d - 1$  neighbors in  $\Gamma_P$ . We consider two cases.

**Case 1 :** Each vertex in  $\Gamma_P$  has exactly  $d - 1$  neighbors in  $\Gamma_P$ .

Observe that by symmetry each vertex of  $\Gamma_{\tilde{P}}$  is also  $(d - 1)$ -valent in  $\Gamma_{\tilde{P}}$  and that the two subgraphs  $\Gamma_P$  and  $\Gamma_{\tilde{P}}$  are copies of each other. We also observe that:

1.  $\Gamma_P$  is  $(d - 1)$ -regular.
2. If  $\Gamma_P$  has more than one connected component, pick one and call it  $C$ . Then the subgraph  $\Gamma_C$  of  $\Gamma$  induced by  $V(C) \cup V(\tilde{C})$  is  $d$ -regular and hence not connected to  $\Gamma \setminus \Gamma_C$  contrary to the assumption that  $\Gamma$  is connected. Hence  $\Gamma_P$  must be connected.
3. Suppose  $x, y \in V(P)$  are separated by  $\Gamma_P$ . Let  $C(x)$  and  $C(y)$  be the connected components of  $G(P) \setminus \Gamma_P$  containing  $x$  and  $y$  respectively. It is easy to see that  $\tilde{C}(x)$  and  $\tilde{C}(y)$  are separated by  $\Gamma_{\tilde{P}}$  in  $G(\tilde{P})$ . Then  $\Gamma$  would separate  $C(x)$  and  $C(y)$ , contrary to our assumption. Hence  $\Gamma_P$  cannot separate  $G(P)$ .

Since Perles' conjecture is true for  $P$ , using 1, 2 and 3 we conclude that  $\Gamma_P$  determines a facet  $F$  of  $P$ . Since  $\Gamma_{\tilde{P}}$  is the image of  $\Gamma_P$  it also determines the facet  $\tilde{F}$  of  $\tilde{P}$ . So  $\Gamma$  determines a facet of  $Pr(P)$ .

**Case 2 :** At least one vertex in  $\Gamma_P$  has  $d$  neighbors in  $\Gamma_P$ .

Let  $X$  be the set of all the  $d$ -valent vertices in  $\Gamma_P$ , i.e.,

$$X = \{w \mid w \in V(\Gamma_P) \text{ and } w \text{ has } d \text{ neighbors in } \Gamma_P\}$$

Let  $Y$  be the set of all vertices in  $\Gamma_P$  that are adjacent to at least one vertex in  $X$ , i.e.,

$$Y = \{w \mid w \notin X; \exists x \in X, (w, x) \in E(\Gamma_P)\}$$

Since vertices in  $Y$  are  $(d-1)$ -valent in  $\Gamma_P$ ,  $\tilde{Y} \subset V(\Gamma_{\tilde{P}})$ . In  $G(\tilde{P})$ , all edges coming out of  $\tilde{X}$  terminate in  $\tilde{Y}$ . In other words, any edge path in  $G(Pr(P))$  between a vertex  $x \in \tilde{X}$  and a vertex  $v \notin \tilde{X}$  must contain a vertex in  $\tilde{Y}$ . We know that there is a vertex  $v \in (G(P) \setminus \Gamma)$  because  $G(P)$  being  $d$ -regular cannot be a proper subgraph of  $\Gamma$  which is also  $d$ -regular (recall that we assumed  $\Gamma \cap V(\tilde{P}) \neq \emptyset$ .) So  $\tilde{Y} \subset V(\Gamma)$  separates  $v$  and  $\tilde{X}$  contrary to the assumption that  $\Gamma$  does not separate  $G(Pr(P))$ . Hence Case 2 is impossible.

The above argument, shows that  $Pr(P)$  satisfies Perles' conjecture if  $P$  does.  $\diamond$

**Theorem 2** *If Perles' conjecture is true for a simple  $d$ -polytope  $P$ , then it is also true for the  $d$ -polytope  $Tr(P)$  obtained by truncating a vertex of  $P$ .*

**Proof :** Assume that vertex  $v \in V(P)$  was truncated to obtain  $Tr(P)$ . Suppose  $(v, w_1), \dots, (v, w_d)$  are the  $d$  edges incident on  $v$  in  $P$ . Then  $z_1, \dots, z_d$  are the new vertices in  $Tr(P)$  where  $z_i$  is the intersection of  $(v, w_i)$  and the truncating hyperplane  $H$ . Also, the new facet of  $Tr(P)$  (namely  $H \cap Tr(P)$ ) is a  $(d-1)$ -simplex determined by the vertex set  $Z = \{z_1, \dots, z_d\}$ .

Let  $\Gamma$  be a  $(d-1)$ -regular, connected, induced and non-separating subgraph of  $G(Tr(P))$ .

If  $Z \cap V(\Gamma) = \emptyset$ , there is nothing to prove. Also, if  $Z \subseteq V(\Gamma)$  then since  $Z$  induces a  $(d-1)$ -regular subgraph  $Z = V(\Gamma)$  and hence  $\Gamma$  determines a facet of

$Tr(P)$ . So the only case left to consider is where  $\Gamma$  contains a proper nonempty subset of  $Z$ . Since  $\Gamma$  is  $(d-1)$ -regular, if it contains a vertex of  $Z$ , it must contain at least  $d-2$  of its neighbors in  $Z$ . Therefore at most one vertex of  $Z$  can be left out and without loss of generality we assume  $z_d \notin V(\Gamma)$ . Hence,  $w_i \in V(\Gamma)$  for  $1 \leq i \leq d-1$ .

Consider the subgraph  $\Gamma'$  of  $G(P)$  induced by the vertex set  $(V(\Gamma) \setminus Z) \cup \{v\}$ .  $\Gamma'$  is a  $(d-1)$ -regular, connected, induced subgraph.  $z_d$  has only one neighbour in  $V(Tr(P)) \setminus V(\Gamma)$ . Therefore  $V(\Gamma) \cup \{z_d\}$  does not separate  $G(Tr(P))$  which means  $\Gamma'$  does not separate  $G(P)$ . So  $\Gamma'$  determines a facet  $F$  of  $P$  and  $w_1, \dots, w_{d-1}$  are the neighbors of  $v$  in  $F$ . Let  $H$  be the supporting hyperplane for  $F$  in  $P$ .  $H \cap Tr(P)$  is a facet of  $Tr(P)$  and the graph of this facet is  $\Gamma$ ; that completes the proof.  $\diamond$

As an immediate consequence of theorems 1 and 2 we have,

**Corollary 1** *Perles' conjecture is true for every polytope in the subclass  $\mathcal{C}$ .*

The subclass  $\mathcal{C}$  has the property that any face of a polytope in  $\mathcal{C}$  also belongs to  $\mathcal{C}$ . We prove this property in the following lemma.

**Lemma 1** *If  $Q \in \mathcal{C}$  and  $F$  is a facet of  $Q$  then  $F \in \mathcal{C}$ .*

**Proof :** Let  $Q \in \mathcal{C}$  be a polytope for which the lemma is true. Let  $X$  be a facet of  $Pr(Q)$ . If  $X = Q$  or  $X = \tilde{Q}$  then by assumption  $X \in \mathcal{C}$ . If however  $X = Pr(F)$  where  $F$  is a facet of  $Q$ , then since the lemma is true for  $Q$ ,  $F \in \mathcal{C}$  and hence  $Pr(F) = X \in \mathcal{C}$ . So, if the lemma is true for a  $Q \in \mathcal{C}$  then it is also true for  $Pr(Q)$ .

Now we consider  $Tr(Q)$ . Let  $v \in V(Q)$  be truncated to obtain  $Tr(Q)$  and let  $H$  be the truncating hyperplane. Assume that  $H^+$  contains  $Tr(Q)$  ( $H^+$  denotes one of the closed half-spaces of  $H$ ). Let  $Y$  be a facet of  $Tr(Q)$ .

If  $Y = H \cap Tr(Q)$  then  $Y$  is a simplex and hence  $Y \in \mathcal{C}$ . So assume  $Y = F \cap H^+$  where  $F$  is a facet of  $Q$ . Since the lemma is true for  $Q$ ,  $F \in \mathcal{C}$ . Hence  $Tr(F) = Y \in \mathcal{C}$ . The only case left to consider is when  $Y = F$  where  $F$  is a facet of  $Q$ . Once again  $Y \in \mathcal{C}$ .

Therefore if the lemma is true for a  $Q \in \mathcal{C}$  it is also true for  $Tr(Q)$ ; that completes the proof.  $\diamond$

As an immediate consequence of this lemma we obtain

**Corollary 2** *The entire combinatorial structure of a polytope  $P \in \mathcal{C}$  can be determined from  $G(P)$  by repeated application of theorems 1 and 2 and lemma 1.*

The following lemma shows that  $\mathcal{C}$  is properly contained in the class of simple polytopes.

**Lemma 2**  *$\mathcal{C}$  is a proper subclass of simple polytopes.*

**Proof :** We show that  $\mathcal{C}$  does not contain a simple 4-polytope with 9 vertices. Suppose it did. Then, to construct the polytope we can either start with a simple 3-polytope or a 4-simplex. Suppose we started with a 4-simplex which has 5 vertices. In this case we may only truncate vertices. But each truncation (when  $d=4$ ) increases the vertex count by 3; so we get 4-polytopes with 5, 8, 11,  $\dots$  vertices but not with 9 vertices. On the contrary suppose we started with a 3-polytope. Since constructing a prism doubles the vertex count we can only construct a prism over a 3-polytope with 4 vertices. The same argument as before shows that again we cannot obtain a 4-polytope with 9 vertices.

Now consider  $C(6,4)$  - the cyclic 4-polytope with 6 vertices has 9 facets. (refer to [3] for details). It is a simplicial polytope. Its dual which is simple has 9 vertices and is hence not in  $\mathcal{C}$ .  $\diamond$

Also, it is easy to show that the dual-stacked and the dual-crosspolytopes form proper subclasses of  $\mathcal{C}$ .

## 4 Remarks

Perles' conjecture is true for any simple 3-polytope[5]. So we could as well start with any simple 3-polytope and build prisms and truncate vertices finitely many



times. The foregoing results would still be valid without any modification for a polytope so obtained.

## References

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